

QUANTUM DISSIPATION AND COHERENCE

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ABSTRACT

We discuss dissipative systems in Quantum Field Theory by studying the canonical quantization of the damped harmonic oscillator (dho). We show that the set of states of the system splits into unitarily inequivalent representations of the canonical commutation relations. The irreversibility of time evolution is expressed as tunneling among the unitarily inequivalent representations. Canonical quantization is shown to lead to time dependent $SU(1,1)$ coherent states. We derive the exact action for the dho from the path integral formulation of the quantum Brownian motion developed by Schwinger and by Feynman and Vernon. The doubling of the phase-space degrees of freedom for dissipative systems is related to quantum noise effects. Finally, we express the time evolution generator of the dho in terms of operators of the q -deformation of the Weyl-Heisenberg algebra. The q -parameter acts as a label for the unitarily inequivalent representations.

1. Introduction

In this paper we want to report on some recent work¹⁻⁵ on dissipative systems in quantum theory.

In principle there is no room for dissipative systems in Quantum Mechanics (QM) since the formalism of QM is based on conserving probabilities, and one has to introduce some sort of generalized quantum formalism to describe damped systems. The developments of the theory of metastable states going beyond the Breit-Wigner treatment and other phenomenological approaches have been recently reported in ref. 6 where the generalized quantum theory for unstable systems of Sudarshan et al.^{6,7} has been also reviewed.

Dissipative systems from the point of view of the quantum theory for Brownian motion have been analyzed in the path integral formalism by Schwinger⁸ and by Feynman and Vernon⁹ and are of course a major topic in nonequilibrium statistical mechanics and nonequilibrium Quantum Field Theory (QFT) at finite temperature^{10,11}.

The microscopic theory for a dissipative system must include the details of processes responsible for dissipation, including quantum effects. One may start since the beginning with a Hamiltonian that describes the system, the bath and the system-bath interaction. Subsequently, the description of the original dissipative system is recovered by the reduced density matrix obtained by eliminating the bath variables which originate the damping and the fluctuations. The problem with

dissipative systems in QM is indeed that canonical commutation relations (ccr) are not preserved by time evolution due to damping terms. The rôle of fluctuating forces is in fact the one of preserving the canonical structure.

At a classical level, it is known since long time¹² that the attempt to derive, from a variational principle, the equations of motion defining the dissipative system requires the introduction of additional complementary equations.

This latter approach has been pursued in refs. 13 and 1-5 where the quantization of the damped harmonic oscillator (dho) has been studied by doubling the phase-space degrees of freedom. The doubled degrees of freedom play the rôle of the bath degrees of freedom. In sec. 2 we present such an approach.

We have shown in refs. 1-3 that the dynamical group structure associated with the canonical quantization of dho is that of $SU(1,1)$ and that time evolution would lead out of the Hilbert space of states; in other words, the quantum mechanical treatment of dho does not provide a unitary irreducible representation of $SU(1,1)$ ¹⁴. To cure these pathologies one must move to QFT, where infinitely many unitarily inequivalent (*ui*) representations of the ccr are allowed (in the infinite volume or thermodynamic limit). The reason for this is that the set of the states of the damped oscillator splits into *ui* representations (i.e. into disjoint *folia*, in the C^* -algebra formalism) each one representing the states of the system at time t : in a more conventional language, the time evolution may be described as *tunneling* between *ui* representations. A remarkable feature of our description thus emerges: *at microscopic level the irreversibility of time evolution (the arrow of time) of damped oscillator is expressed by the non unitary evolution across the ui representations of the ccr.*

We stress that the nature of the ground states of the *ui* representations is the one of the $SU(1,1)$ coherent states. Furthermore, it has been shown¹ that the squeezed coherent states of light entering quantum optics¹⁵ can be identified, up to elements of the group \mathcal{G} of automorphisms of $su(1,1)$, with the states of the quantum dho.

In ref. 3 it has been shown that the dho states are time dependent thermal states, as expected due to the statistical nature of dissipation. The formalism for the dho turns out to be similar to the one of real time QFT at finite temperature, also called thermo-field dynamics (TFD)^{11,16,17}. In refs. 18 and 19 such a connection with TFD has been further analysed and the master equation has been discussed¹⁸.

In ref. 4 the exact action for the dho in the path integral formalism of Schwinger and Feynman and Vernon has been obtained. In particular the initial values of the doubled variables have been related to the probability of quantum fluctuations in the ground state, a result which is interesting also in the more general case of thermal field theories. We report such results in sec. 3.

Finally, in sec. 4, we explicitly express⁵ the time evolution generator of the dho in terms of operators of q -deformed Weyl-Heisenberg (q -WH)²⁰ algebra. The relation of q -WH algebra with thermal field theory is also commented upon. The q -deformation parameter turns out to be related with time parameter in the case of dho and with temperature in the case of thermal field theory. In both cases, the q -parameter acts as a label for the *ui* representations of the ccr in which the

space of the states splits in the infinite volume limit. Such a conclusion confirms a general analysis ²¹ which shows that the Weyl representations in QM and the *ui* representations in QFT are indeed labelled by the deformation parameter of the q-WH algebra. Sec. 5 is devoted to the conclusions.

2. The canonical quantization of the damped harmonic oscillator

We consider the damped harmonic oscillator with classical equation

$$m\ddot{x} + \gamma\dot{x} + \kappa x = 0 \quad , \quad (2.1)$$

and we want to perform its canonical quantization. We closely follow the approach of refs. 1-3 and 13.

In order to deal with an isolated system, as the canonical quantization scheme requires, it is necessary to double the the phase-space dimensions^{12,13}. The lagrangian for system 2.1 is written as

$$L = m\dot{x}\dot{y} + \frac{1}{2}\gamma(x\dot{y} - \dot{x}y) - \kappa xy \quad . \quad (2.2)$$

Eq.2.1 is obtained by varying eq. 2.2 with respect to y , whereas variation with respect to x gives

$$m\ddot{y} - \gamma\dot{y} + \kappa y = 0 \quad , \quad (2.3)$$

which appears to be the *time reversed* ($\gamma \rightarrow -\gamma$) of eq. 2.1. y may be thought of as describing an effective degree of freedom for the heat bath to which the system 2.1 is coupled. The canonical momenta are then given by $p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{1}{2}\gamma y$; $p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{1}{2}\gamma x$. The hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{m}p_x p_y + \frac{1}{2m}\gamma(y p_y - x p_x) + \left(\kappa - \frac{\gamma^2}{4m}\right) xy \quad . \quad (2.4)$$

Canonical quantization is then performed by introducing the commutators $[x, p_x] = i\hbar = [y, p_y]$, $[x, y] = 0 = [p_x, p_y]$, and the corresponding sets of annihilation and creation operators

$$\begin{aligned} \alpha &\equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right) \quad , \quad \alpha^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right) \quad , \\ \beta &\equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right) \quad , \quad \beta^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{\frac{1}{2}} \left(\frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y\right) \quad , \end{aligned} \quad (2.5a)$$

$$[\alpha, \alpha^\dagger] = 1 = [\beta, \beta^\dagger] \quad , \quad [\alpha, \beta] = 0 = [\alpha, \beta^\dagger] \quad . \quad (2.5b)$$

We have introduced $\Omega \equiv \left[\frac{1}{m}\left(\kappa - \frac{\gamma^2}{4m}\right)\right]^{\frac{1}{2}}$, the common frequency of the two oscillators eq. 2.1 and eq. 2.3, assuming Ω real, hence $\kappa > \frac{\gamma^2}{4m}$ (case of no overdamping). The Feshbach and Tikochinsky ¹³ quantum hamiltonian is then obtained as

$$H = \hbar\Omega(\alpha^\dagger\beta + \alpha\beta^\dagger) - \frac{i\hbar\gamma}{4m}[(\alpha^2 - \alpha^{\dagger 2}) - (\beta^2 - \beta^{\dagger 2})] \quad . \quad (2.6)$$

In sec. 3 we show that, at quantum level, the β modes allow quantum noise effects arising from the imaginary part of the action⁴. By using the canonical linear transformations $A \equiv \frac{1}{\sqrt{2}}(\alpha + \beta)$, $B \equiv \frac{1}{\sqrt{2}}(\alpha - \beta)$, H is written as

$$\begin{aligned} H &= H_0 + H_I \quad , \\ H_0 &= \hbar\Omega(A^\dagger A - B^\dagger B) \quad , \quad H_I = i\hbar\Gamma(A^\dagger B^\dagger - AB) \quad , \end{aligned} \quad (2.7)$$

where the decay constant for the classical variable $x(t)$ is denoted by $\Gamma \equiv \frac{\gamma}{2m}$.

We note that the states generated by B^\dagger represent the sink where the energy dissipated by the quantum damped oscillator flows: the B -oscillator represents the reservoir or heat bath coupled to the A -oscillator.

The dynamical group structure associated with the system of coupled quantum oscillators is that of $SU(1, 1)$. The two mode realization of the algebra $su(1, 1)$ is indeed generated by $J_+ = A^\dagger B^\dagger$, $J_- = J_+^\dagger = AB$, $J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1)$, $[J_+, J_-] = -2J_3$, $[J_3, J_\pm] = \pm J_\pm$. The Casimir operator \mathcal{C} is $\mathcal{C}^2 \equiv \frac{1}{4} + J_3^2 - \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{1}{4}(A^\dagger A - B^\dagger B)^2$.

We also observe that $[H_0, H_I] = 0$. The time evolution of the vacuum $|0\rangle \equiv |n_A = 0, n_B = 0\rangle$, $A|0\rangle = 0 = B|0\rangle$, is controlled by H_I

$$\begin{aligned} |0(t)\rangle &= \exp\left(-it\frac{H}{\hbar}\right)|0\rangle = \exp\left(-it\frac{H_I}{\hbar}\right)|0\rangle \\ &= \frac{1}{\cosh(\Gamma t)} \exp(\tanh(\Gamma t)A^\dagger B^\dagger)|0\rangle \quad , \end{aligned} \quad (2.8a)$$

$$\langle 0(t)|0(t)\rangle = 1 \quad \forall t \quad , \quad (2.8b)$$

$$\lim_{t \rightarrow \infty} \langle 0(t)|0\rangle \propto \lim_{t \rightarrow \infty} \exp(-t\Gamma) = 0 \quad . \quad (2.9)$$

Time evolution transformations for creation and annihilation operators are

$$\alpha \mapsto \alpha(t) = e^{-i\frac{t}{\hbar}H_I} \alpha e^{i\frac{t}{\hbar}H_I} = \alpha \cosh(\Gamma t) - \alpha^\dagger \sinh(\Gamma t) \quad , \quad (2.10a)$$

$$\beta \mapsto \beta(t) = e^{-i\frac{t}{\hbar}H_I} \beta e^{i\frac{t}{\hbar}H_I} = \beta \cosh(\Gamma t) + \beta^\dagger \sinh(\Gamma t) \quad (2.10b)$$

and h.c., and the corresponding ones for $A(t)$, $B(t)$ and h.c.. We note that eqs. 2.10 are canonical transformations preserving the ccr eq. 2.5b. Eq.2.9 expresses the instability (decay) of the vacuum under the evolution operator $\exp(-it\frac{H_I}{\hbar})$. This means that the QM framework is not suitable for the canonical quantization of the dho. In other words time evolution leads out of the Hilbert space of the states and in ref. 3 it has been shown that the proper way to perform the canonical quantization of the dho is to work in the framework of QFT. In fact for many degrees of freedom the time evolution operator $\mathcal{U}(t)$ and the vacuum are formally (at finite volume) given by

$$\begin{aligned} \mathcal{U}(t) &= \prod_{\kappa} \exp\left(-\frac{\Gamma_{\kappa} t}{2}(\alpha_{\kappa}^2 - \alpha_{\kappa}^{\dagger 2})\right) \exp\left(\frac{\Gamma_{\kappa} t}{2}(\beta_{\kappa}^2 - \beta_{\kappa}^{\dagger 2})\right) \\ &= \prod_{\kappa} \exp\left(\Gamma_{\kappa} t(A_{\kappa}^\dagger B_{\kappa}^\dagger - A_{\kappa} B_{\kappa})\right), \end{aligned} \quad (2.11)$$

$$|0(t)\rangle = \prod_{\kappa} \frac{1}{\cosh(\Gamma_{\kappa} t)} \exp(\tanh(\Gamma_{\kappa} t) A_{\kappa}^{\dagger} B_{\kappa}^{\dagger}) |0\rangle, \quad (2.12)$$

with $\langle 0(t)|0(t)\rangle = 1$, $\forall t$. Using the continuous limit relation $\sum_{\kappa} \mapsto \frac{V}{(2\pi)^3} \int d^3\kappa$, in the infinite-volume limit we have (for $\int d^3\kappa \Gamma_{\kappa}$ finite and positive)

$$\langle 0(t)|0\rangle \rightarrow 0 \text{ as } V \rightarrow \infty \quad \forall t, \quad (2.13)$$

and in general, $\langle 0(t)|0(t')\rangle \rightarrow 0$ as $V \rightarrow \infty \quad \forall t \text{ and } t', \quad t' \neq t$. At each time t a representation $\{|0(t)\rangle\}$ of the ccr is defined and turns out to be *ui* to any other representation $\{|0(t')\rangle, \quad \forall t' \neq t\}$ in the infinite volume limit. In such a way the quantum dho evolves in time through *ui* representations of ccr (*tunneling*). We remark that $|0(t)\rangle$ is a two-mode time dependent Glauber coherent state^{22,23}.

We thus see that the Bogolubov transformations, corresponding to eqs.2.10, can be implemented for every κ as inner automorphism for the algebra $su(1,1)_{\kappa}$. At each time t we have a copy $\{A_{\kappa}(t), A_{\kappa}^{\dagger}(t), B_{\kappa}(t), B_{\kappa}^{\dagger}(t); |0(t)\rangle \equiv |\nu_{\kappa}\rangle\}$ of the original algebra induced by the time evolution operator which can thus be thought of as a generator of the group of automorphisms of $\bigoplus_{\kappa} su(1,1)_{\kappa}$ parameterized by time t (we have a realization of the operator algebra at each time t , which can be implemented by Gel'fand-Naimark-Segal construction in the C^* -algebra formalism¹⁰). Notice that the various copies become unitarily inequivalent in the infinite-volume limit, as shown by eqs.2.13: the space of the states splits into *ui* representations of the ccr each one labelled by time parameter t . As usual one works at finite volume and only at the end of the computations the limit $V \rightarrow \infty$ is performed.

Finally, in refs. 2 and 3 it has been shown that the representation $\{|0(t)\rangle\}$ is equivalent to the TFD representation $\{|0(\beta(t))\rangle\}$, thus recognizing the relation between the dho states and the finite temperature states. In particular, one may introduce the *free energy* functional for the A -modes

$$\mathcal{F}_A \equiv \langle 0(t) | \left(H_A - \frac{1}{\beta} S_A \right) | 0(t) \rangle, \quad (2.14)$$

where H_A is the part of H_0 relative to A -modes only, namely $H_A = \sum_{\kappa} \hbar \Omega_{\kappa} A_{\kappa}^{\dagger} A_{\kappa}$, and the *entropy* S_A is given by

$$S_A \equiv - \sum_{\kappa} \left\{ A_{\kappa}^{\dagger} A_{\kappa} \ln \sinh^2(\Gamma_{\kappa} t) - A_{\kappa} A_{\kappa}^{\dagger} \ln \cosh^2(\Gamma_{\kappa} t) \right\}. \quad (2.15)$$

One then considers the stability condition $\frac{\partial \mathcal{F}_A}{\partial \vartheta_{\kappa}} = 0 \quad \forall \kappa$, $\vartheta_{\kappa} \equiv \Gamma_{\kappa} t$ to be satisfied in each representation, and using the definition $E_{\kappa} \equiv \hbar \Omega_{\kappa}$, one finds

$$\mathcal{N}_{A_{\kappa}}(t) = \sinh^2(\Gamma_{\kappa} t) = \frac{1}{e^{\beta(t) E_{\kappa}} - 1}, \quad (2.16)$$

which is the Bose distribution for A_{κ} at time t . $\{|0(t)\rangle\}$ is thus recognized to be a representation of the ccr's at finite temperature, equivalent to the TFD representation $\{|0(\beta)\rangle\}$ ^{11,16,17}. Furthermore, use of eq.2.15 shows that

$$\frac{\partial}{\partial t} |0(t)\rangle = - \left(\frac{1}{2} \frac{\partial \mathcal{S}}{\partial t} \right) |0(t)\rangle. \quad (2.17)$$

We thus see that $i(\frac{1}{2}\hbar\frac{\partial\mathcal{S}}{\partial t})$ is the generator of time translations, namely time evolution is controlled by the entropy variations²⁴. It appears to us remarkable that the same dynamical variable \mathcal{S} whose expectation value is formally the entropy also controls time evolution: Damping (or, more generally, dissipation) implies indeed the choice of a privileged direction in time evolution (*arrow of time*) with a consequent breaking of time-reversal invariance. We may also show that $d\mathcal{F}_A = dE_A - \frac{1}{\beta}d\mathcal{S}_A = 0$, which expresses the first principle of thermodynamics for a system coupled with environment at constant temperature and in absence of mechanical work. We may define as usual heat as $dQ = \frac{1}{\beta}d\mathcal{S}$ and see that the change in time $d\mathcal{N}_A$ of particles condensed in the vacuum turns out into heat dissipation dQ .

3. Quantum dissipation and quantum noise

Let us now ask the following question: Does the introduction of an “extra coordinate” make any sense in the context of conventional QM? To answer to such a question we consider the special case of zero mechanical resistance. One then begins with the Hamiltonian for an isolated particle and the corresponding density matrix equation

$$H = -(\hbar^2/2m)(\partial/\partial Q)^2 + V(Q). \quad (3.1)$$

$$i\hbar(\partial\rho/\partial t) = [H, \rho], \quad (3.2)$$

which indeed requires two coordinates (say Q_+ and Q_-). In the coordinate representation, we have⁴

$$i\hbar(\partial/\partial t) < Q_+ | \rho(t) | Q_- > = \{ -(\hbar^2/2m)[(\partial/\partial Q_+)^2 - (\partial/\partial Q_-)^2] + [V(Q_+) - V(Q_-)] \} < Q_+ | \rho(t) | Q_- >. \quad (3.3)$$

In terms of the coordinates x and y , it is $Q_{\pm} = x \pm (1/2)y$, and the density matrix function $W(x, y, t) = < x + (1/2)y | \rho(t) | x - (1/2)y >$. From eq.3.3 the Hamiltonian now reads $\mathcal{H}_o = (p_x p_y / m) + V(x + (1/2)y) - V(x - (1/2)y)$, with $p_x = -i\hbar(\partial/\partial x)$, $p_y = -i\hbar(\partial/\partial y)$, which, of course, may be constructed from the “Lagrangian”

$$\mathcal{L}_0(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} - V(x + (1/2)y) + V(x - (1/2)y), \quad (3.4)$$

We have then the justification for introducing eq.2.2 at least for the case $\gamma = 0$. Notice indeed that for $V(x \pm (1/2)y) = (1/2)k(x \pm (1/2)y)$ eq.3.4 gives eq.2.2 for the case $\gamma = 0$. We also notice that \mathcal{H}_o and \mathcal{H}_I in eq.2.7 are the free Hamiltonian and the generator of Bogolubov transformations, respectively, in TFD^{11,16,17}. Our present discussion thus includes the doubling of degrees of freedom in finite temperature QFT.

Next, our task is to explore the manner in which the Lagrangian model for quantum dissipation of refs. 1-5,13 arises from the formulation of the quantum Brownian motion problem as described by Schwinger⁸ and by Feynman and Vernon⁹.

Let us suppose that the particle interacts with a thermal bath at temperature T . The interaction Hamiltonian between the bath and the particle is taken as $H_{int} = -fQ$, where Q is the particle coordinate and f is the random force on the

particle due to the bath. In the Feynman-Vernon formalism the effective action for the particle has the form

$$\mathcal{A}[x, y] = \int_{t_i}^{t_f} dt \mathcal{L}_o(\dot{x}, \dot{y}, x, y) + \mathcal{I}[x, y], \quad (3.5)$$

where \mathcal{L}_o is defined in eq.3.4 and

$$e^{(i/\hbar)\mathcal{I}[x, y]} = \langle (e^{(-i/\hbar) \int_{t_i}^{t_f} f(t)Q_-(t)dt})_- (e^{(i/\hbar) \int_{t_i}^{t_f} f(t)Q_+(t)dt})_+ \rangle. \quad (3.6)$$

In eq.3.6 the average is with respect to the thermal bath; “ $(\cdot)_+$ ” denotes time ordering and “ $(\cdot)_-$ ” denotes anti-time ordering. If the interaction between the bath and the coordinate Q were turned off, then the operator f of the bath would develop in time according to $f(t) = e^{iH_R t/\hbar} f e^{-iH_R t/\hbar}$ where H_R is the Hamiltonian of the isolated bath (decoupled from the coordinate Q). $f(t)$ is the force operator of the bath to be used in eq.3.6. Assuming that the particle makes contact with the bath at the initial time t_i , the reduced density matrix function is at a final time

$$W(x_f, y_f, t_f) = \int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dy_i K(x_f, y_f, t_f; x_i, y_i, t_i) W(x_i, y_i, t_i), \quad (3.7)$$

$$K(x_f, y_f, t_f; x_i, y_i, t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t) \int_{y(t_i)=y_i}^{y(t_f)=y_f} \mathcal{D}y(t) e^{(i/\hbar)\mathcal{A}[x, y]}. \quad (3.8)$$

The correlation function for the random force on the particle is given by $G(t-s) = (i/\hbar) \langle f(t)f(s) \rangle$. The retarded and advanced Greens functions are defined by $G_{ret}(t-s) = \theta(t-s)[G(t-s) - G(s-t)]$, and $G_{adv}(t-s) = \theta(s-t)[G(s-t) - G(t-s)]$. The mechanical resistance is defined $R = \lim_{\omega \rightarrow 0} \text{Re} Z(\omega + i0^+)$, with the mechanical impedance $Z(\zeta)$ (analytic in the upper half complex frequency plane $\text{Im } \zeta > 0$) determined by the retarded Greens function $-i\zeta Z(\zeta) = \int_0^\infty dt G_{ret}(t) e^{i\zeta t}$. The time domain quantum noise in the fluctuating random force is $N(t-s) = (1/2) \langle f(t)f(s) + f(s)f(t) \rangle$.

The time ordered and anti-time ordered Greens functions describe both the retarded and advanced Greens functions as well as the quantum noise,

$$G_\pm(t-s) = \pm(1/2)[G_{ret}(t-s) + G_{adv}(t-s)] + (i/\hbar)N(t-s). \quad (3.9)$$

The interaction between the bath and the particle is evaluated by following Feynman and Vernon and we find⁴ for the real and the imaginary part of the action

$$\text{Re} \mathcal{A}[x, y] = \int_{t_i}^{t_f} dt \mathcal{L}, \quad (3.10a)$$

$$\mathcal{L} = m\dot{x}\dot{y} - [V(x + (1/2)y) - V(x - (1/2)y)] + (1/2)[xF_y^{ret} + yF_x^{adv}], \quad (3.10b)$$

$$\text{Im} \mathcal{A}[x, y] = (1/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s) y(t) y(s), \quad (3.10c)$$

respectively, where the retarded force on y and the advanced force on x are defined as $F_y^{ret}(t) = \int_{t_i}^{t_f} ds G_{ret}(t-s) y(s)$, $F_x^{adv}(t) = \int_{t_i}^{t_f} ds G_{adv}(t-s) x(s)$.

Eqs.3.10 are *rigorously exact* for linear passive damping due to the bath when the path integral eq.3.8 is employed for the time development of the density matrix.

We therefore conclude that the lagrangian eq.2.2 can be viewed as the approximation to eq.3.10b with $F_y^{ret} = \gamma \dot{y}$ and $F_x^{adv} = -\gamma \dot{x}$.

We also observe that at the classical level the “extra” coordinate y , is usually constrained to vanish. (Note that $y(t) = 0$ is a true solution to eqs.2.2 so that the constraint is *not* in violation of the equations of motion.) From eqs.3.10 we thus also conclude that the classical constraint $y = 0$ occurs because nonzero y yields an “unlikely process” in view of the large imaginary part of the action (in the classical “ $\hbar \rightarrow 0$ ” limit) implicit in eq.3.10c. On the contrary, at quantum level nonzero y allows quantum noise effects arising from the imaginary part of the action.

4. Quantum dissipation and the q-deformation of the WH algebra

Quantum deformations^{20,25} of Lie algebras are well studied mathematical structures and therefore their properties need not to be presented again in this paper; we only recall that they are deformations of the enveloping algebras of Lie algebras and have Hopf algebra structure. In particular the q-WH algebra has the properties of graded Hopf algebra²⁶. In this section our task is to establish a formal relation between the q-WH algebra and the dho hamiltonian. To this aim we introduce the realization of q-WH algebra in terms of finite difference operators²⁷. Since we want to preserve the analytic properties of Lie algebra in the deformation procedure, we work in the Fock-Bargmann representation (FBR) in QM²³. In the FBR the operators $N \rightarrow z \frac{d}{dz}$, $a^\dagger \rightarrow z$, $a \rightarrow \frac{d}{dz}$, $z \in \mathbb{C}$, provide a realization of the WH algebra $[a, a^\dagger] = \mathbb{I}$, $[N, a] = -a$, $[N, a^\dagger] = a^\dagger$. The Hilbert space \mathcal{F} is identified with the space of the entire analytic functions and has well defined inner product.

We consider the finite difference operator \mathcal{D}_q defined by: $\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} = \frac{q^z \frac{d}{dz} - 1}{(q-1)} f(z)$, with $f \in \mathcal{F}$, $q = e^\zeta$, $\zeta \in \mathbb{C}$. \mathcal{D}_q is called the q-derivative operator and, for $q \rightarrow 1$ (i.e. $\zeta \rightarrow 0$), it reduces to the standard derivative. Next, we introduce the following operators in the space \mathcal{F} : $N \rightarrow z \frac{d}{dz}$, $\hat{a}_q \rightarrow z$, $a_q \rightarrow \mathcal{D}_q$, where $\hat{a}_q = \hat{a}_{q=1} = a^\dagger$ and $\lim_{q \rightarrow 1} a_q = a$. The q-WH algebra, in terms of the operators $\{a_q, \hat{a}_q, N \equiv N_q; q \in \mathbb{C}\}$, is then realized by the relations:

$$[N, a_q] = -a_q, \quad [N, \hat{a}_q] = \hat{a}_q, \quad [a_q, \hat{a}_q] \equiv a_q \hat{a}_q - \hat{a}_q a_q = q^N. \quad (4.1)$$

By introducing $\bar{a}_q \equiv \hat{a}_q q^{-N/2}$, it assumes the more familiar form²⁰ $[N, a_q] = -a_q$, $[N, \bar{a}_q] = \bar{a}_q$, $a_q \bar{a}_q - q^{-\frac{1}{2}} \bar{a}_q a_q = q^{\frac{1}{2}N}$. We can show that the commutator $[a_q, \hat{a}_q]$ acts in \mathcal{F} as follows²⁷

$$[a_q, \hat{a}_q] f(z) = q^{z \frac{d}{dz}} f(z) = f(qz). \quad (4.2)$$

The q-deformation of the WH algebra is thus strictly related with the finite difference operator \mathcal{D}_q ($q \neq 1$). This leads us to conjecture that the q-deformation of the operator algebra should arise whenever we are in the presence of lattice or

discrete structure²⁷. In the following we show that this indeed happens in the case of damping where the finite life-time $\tau = \frac{1}{\Gamma}$ acts as time-unit for the system.

We now express the time evolution generator of dho in terms of the commutator $[a_q, \hat{a}_q]$ of the q-WH algebra. In the FBR Hilbert space \mathcal{F} we introduce

$$\tilde{a} \equiv \frac{1}{\sqrt{2\hbar\Omega}} \left(\frac{p_z}{\sqrt{m}} - i\sqrt{m}\Omega z \right), \quad \tilde{a}^\dagger \equiv \frac{1}{\sqrt{2\hbar\Omega}} \left(\frac{p_z}{\sqrt{m}} + i\sqrt{m}\Omega z \right), \quad [\tilde{a}, \tilde{a}^\dagger] = \mathbb{I} \quad (4.3)$$

where $z \in \mathbb{C}$, $p_z = -i\hbar \frac{d}{dz}$ and $[z, p_z] = i\hbar$. The conjugation of \tilde{a} and \tilde{a}^\dagger is as usual well defined with respect to the inner product defined in \mathcal{F} ²³.

We note that, by setting $Re(z) = x$, $\tilde{a} \rightarrow \alpha$ and $\tilde{a}^\dagger \rightarrow \alpha^\dagger$ in the limit $Im(z) \rightarrow 0$, where α and α^\dagger are the annihilation and creation operators introduced in eqs.2.5. By putting $q = e^\theta$ with θ real we can check that the operator

$$[a_q, \hat{a}_q] = \exp\left(\theta z \frac{d}{dz}\right) = \frac{1}{\sqrt{q}} \exp\left(-\frac{\theta}{2}(\tilde{a}^2 - \tilde{a}^{\dagger 2})\right) \equiv \frac{1}{\sqrt{q}} \hat{S}(\theta), \quad (4.4)$$

generates the Bogolubov transformations:

$$\begin{aligned} \tilde{a}(\theta) &= \hat{S}(\theta) \tilde{a} \hat{S}(\theta)^{-1} = \tilde{a} \cosh \theta - \tilde{a}^\dagger \sinh \theta \\ \mapsto \alpha(\theta) &= \alpha \cosh \theta - \alpha^\dagger \sinh \theta \quad \text{as } Im(z) \rightarrow 0. \end{aligned} \quad (4.5)$$

and h.c.. Note that the right hand side of eq. 4.4 is an $SU(1,1)$ group element. In fact $\frac{1}{2}\tilde{a}^2 = K_-$, $\frac{1}{2}\tilde{a}^{\dagger 2} = K_+$, $\frac{1}{2}(\tilde{a}^\dagger \tilde{a} + \frac{1}{2}) = K_3$, close the $su(1,1)$ algebra. We remark that the transformation eq. 4.5, which is a canonical transformation, has been shown²¹ to relate the Weyl representations of the ccr in QM²³ and thus the deformation parameter $q = e^\theta$ acts as a label for the Weyl representations.

Next, we introduce the q-WH algebra operators $b_{q'}$ and $\hat{b}_{q'}$ corresponding to the doubled degree of freedom β introduced in sec. 2; by introducing the operators

$$\tilde{b} \equiv \frac{1}{\sqrt{2\hbar\Omega}} \left(\frac{p_\zeta}{\sqrt{m}} - i\sqrt{m}\Omega \zeta \right), \quad \tilde{b}^\dagger \equiv \frac{1}{\sqrt{2\hbar\Omega}} \left(\frac{p_\zeta}{\sqrt{m}} + i\sqrt{m}\Omega \zeta \right), \quad [\tilde{b}, \tilde{b}^\dagger] = \mathbb{I} \quad (4.6)$$

with $\zeta \in \mathbb{C}$, $Re(\zeta) \equiv y$, $p_\zeta = -i\hbar \frac{d}{d\zeta}$ and $[\zeta, p_\zeta] = i\hbar$, we proceed as in the above case of \tilde{a} operator and obtain

$$[a_q, \hat{a}_q][b_{q'}, \hat{b}_{q'}] = \exp\left(-\frac{\theta}{2}[(\tilde{a}^2 - \tilde{a}^{\dagger 2}) - (\tilde{b}^2 - \tilde{b}^{\dagger 2})]\right), \quad (4.7)$$

with $q' = q^{-1}$. We finally see that, provided we set $q = e^\theta$, with $\theta \equiv \Gamma t$, the operator in eq.4.7 acts as the time evolution operator $\mathcal{U}(t)$ (cf. eq.2.11) in the limits $Im(z) \rightarrow 0$ and $Im(\zeta) \rightarrow 0$:

$$[a_q, \hat{a}_q][b_{q'}, \hat{b}_{q'}] \mapsto \exp\left(-\frac{\Gamma t}{2}[(\alpha^2 - \alpha^{\dagger 2}) - (\beta^2 - \beta^{\dagger 2})]\right) = \mathcal{U}(t) \quad (4.8)$$

Eq.4.8 is the wanted expression of the time-evolution generator of the dho in terms of the q-WH algebra operator $[a_q, \hat{a}_q][b_{q'}, \hat{b}_{q'}]$.

Our discussion can be generalized to the case of many degrees of freedom by observing that eq.4.8 holds for each couple of modes $(\alpha_\kappa, \beta_\kappa)$ and formally we have

$$\prod_{\kappa} [a_{\kappa,q}, \hat{a}_{\kappa,q}][b_{\kappa,q'}, \hat{b}_{\kappa,q'}] \rightarrow \exp\left(-\frac{i}{\hbar} H_I t\right) = \mathcal{U}(t) \quad , \quad \text{as } \text{Im}\{z, \zeta\} \rightarrow 0 \quad , \quad (4.9)$$

where $q = e^{\theta_\kappa}$, $\theta_\kappa \equiv \Gamma_\kappa t$, $q' = q^{-1}$, $\text{Im}\{z, \zeta\}$ denotes the imaginary part of FBR z - and ζ -variables associated to each α_κ and β_κ mode and $\mathcal{U}(t)$ is given by eq.2.11. Notice that for simplicity of notation here q denotes q_κ . We note that, through its time dependence, the deformation parameter $q(t)$ labels the *ui* representations $\{|O(t)\rangle\}$.

We also observe that eq.4.9 leads to representation of the generator of the thermal Bogolubov transformation in terms of the operators $\prod_{\kappa} [a_{\kappa,q}, \hat{a}_{\kappa,q}][b_{\kappa,q'}, \hat{b}_{\kappa,q'}]$: indeed, by resorting to the discussion of sec. 2 where the representation $\{|O(t)\rangle\}$ for the dho has been shown to be the same as the TFD representation $\{|O(\beta(t))\rangle\}$ (see refs. 2,3), we also recognize the strict relation between q-WH algebra and finite temperature QFT.

5. Comments and conclusion

The dho total Hamiltonian is invariant under the transformations generated by $J_2 = \bigoplus_{\kappa} J_2^{(\kappa)}$. The vacuum however is not invariant under J_2 (see eq.2.8a) in the infinite volume limit. Moreover, at each time t , the representation $\{|O(t)\rangle\}$ may be characterized by the expectation value in the state $|0(t)\rangle$ of, *e.g.*, $J_3^{(\kappa)} - \frac{1}{2}$: thus the total number of particles $n_A + n_B = 2n$ can be taken as an order parameter. Therefore, at each time t the symmetry under J_2 transformations is spontaneously broken. On the other hand, \mathcal{H}_I is proportional to J_2 . Thus, in addition to breakdown of time-reversal (discrete) symmetry, already mentioned in sec. 2, we also have spontaneous breakdown of time translation (continuous) symmetry. In other words we have described dissipation (*i.e.* energy non-conservation), as an effect of breakdown of time translation and time-reversal symmetry. It is an interesting question asking which is the zero-frequency mode, playing the rôle of the Goldstone mode, related with the breakdown of continuous time translation symmetry: we observe that since $n_A - n_B$ is constant in time, the condensation (annihilation and/or creation) of AB-pairs does not contribute to the vacuum energy so that AB-pair may play the rôle of a zero-frequency mode.

From the point of view of boson condensation, time evolution in the presence of damping may be thought of as a sort of continuous transition among different phases, each phase corresponding, at time t , to the coherent state representation $\{|O(t)\rangle\}$. The damped oscillator thus provides an archetype of system undergoing continuous phase transition.

In the discussion presented above a crucial rôle is played by the existence of infinitely many *ui* representations of the ccr in QFT. In ref. 21 the q-WH algebra has been discussed in relation with the von Neumann theorem in QM and it has been shown on a general ground that the q-deformation parameter acts as a label for the *Weyl systems* in QM and for the *ui* representations in QFT; the mapping between

different (i.e. labelled by different values of q) representations (or Weyl systems) being performed by the Bogolubov transformations (at finite volume). Damped harmonic oscillator and finite temperature systems are explicit examples clarifying the physical meaning of such a labelling (further examples are provided by unstable particles in QFT²⁴, by quantization of the matter field in curved space-time²⁸, by theories with spontaneous breakdown of symmetry where different values of the order parameter are associated to different ui representations (different *phases*)). In the case of damping, as well as in the case of time-dependent temperature, the system time-evolution is represented as *tunneling* through ui representations: the non-unitary character of time-evolution (*arrow of time*) is thus expressed by the non-unitary equivalence of the representations in the infinite volume limit. It is remarkable that at the algebraic level this is made possible through the q -deformation mechanism which organizes the representations in an *ordered set* by means of the labelling.

In ref. 27 it has been also shown that the commutator $[a_q, \hat{a}_q]$ acts as squeezing generator (indeed the operator $\hat{S}(\theta)$ in eq.4.4 acts like the squeezing generator with respect to \tilde{a} and \tilde{a}^\dagger operators), a result which confirms the relation between dissipation and squeezed coherent states exhibited in ref. 1. In turn, q -groups have been also shown²⁹ to be the natural candidates to study squeezed coherent state.

Finally, we mention that dissipative systems have been discussed in the framework of stochastic quantization and coherent states in ref. 30.

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